I.S.I. Bangalore Centre
First Semestral Examination 2000-2001
B. Math. Hons. Algebra I

Answer five questions from among questions 1 to 8 and five questions from among questions 9 to 15. All questions carry equal marks.

1. Let $a_1, \ldots, a_n$ be distinct complex numbers. Show that every complex polynomial $f = c_0 + c_1 X + \ldots + c_{n-1} X^{n-1}$ of degree $< n$ can be written as $\sum_{i=1}^{n} \lambda_i (X + a_i)^{n-1}$ for some λ_i, by considering the system of linear equations in the variables λ_i obtained by equating coefficients of corresponding powers of X.

2. (i) Define a permutation matrix and prove that its transpose is its inverse.

   (ii) Find the permutation matrix corresponding to the permutation σ of $\{1, 2, \ldots, n\}$ which interchanges i and $n - i + 1$ for $i = 1, \ldots, n$.

3. Define a linearly independent set B in any vector space V over a field F. If B is a maximal, linearly independent subset of V, prove that B is a basis of V.

4. Given a linear transformation $T : V \to W$ of finite-dimensional vector spaces V, W over a field F, define the matrix A representing T with respect to ordered bases B and B' of V, W respectively. Determine the effect on A when B and B' are replaced by another pair of bases.

5. For $A \in M_n(\mathbf{C})$, define the trace as $tr(A) = \sum_{i=1}^{n} a_{ii}$.

   (i) Prove that $tr(AB) = tr(BA) \ \forall \ A, B \in M_n(\mathbf{C})$. Deduce that $tr(PAP^{-1}) = tr(A) \ \forall \ P \in GL_n(\mathbf{C})$.

   (ii) Show that $tr(A)$ is the sum of all eigenvalues of A.

6. Let W be a subspace of a finite-dimensional vector space V over a field F. Using (i.e., assuming) the fact that a basis of W can be extended to a basis of V, prove that $\dim V/W = \dim V - \dim W$.

7. (i) Find the eigenvalues of $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and further find $P \in GL_2(\mathbf{C})$ such that PAP^{-1} is diagonal.

   (ii) Show that a matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with a, b, c real has real eigenvalues.

8. For a matrix $A \in M_n(\mathbb{R})$, show that the following are equivalent:

   (i) A is orthogonal i.e. $A^t = A^{-1}$.

   (ii) $\langle Av, Aw\rangle = \langle v, w\rangle \ \forall \ v, w$ where this is the usual dot product on $\mathbb{R}^n$.

   (iii) The columns of A form a basis of $\mathbb{R}^n$ consisting of unit vectors which are mutually orthogonal.

9. Let G be a finite group and let H be a subgroup. Prove that $O(H)$ divides $O(G)$. Deduce that, for each $g \in G, O(g)/O(G)$.

10. (i) Define a normal subgroup of a group G. Write down all subgroups of S_3 and find out which ones are normal.

    (ii) Prove that $(1\ 2)$ and $(1\ 2 \ldots n)$ generate S_n.

11. For a group G, define the commutator subgroup $[G, G]$. Further, prove that $[G, G]$ is a normal subgroup of G and that $G/[G, G]$ is an abelian group.

12. (i) Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if $(m, n) = 1$.

    (ii) Show that a group of order p^2 (where p is a prime) is abelian.

13. If $O(G) = pq$ where $p \neq q$ are primes and, if P, Q are normal subgroups of order p and q respectively, show that G is cyclic.

14. Let $H \neq G$ be a subgroup of a finite group G. Prove that $G \neq \bigcup\limits_{g \in G} gHg^{-1}$.

15. For an element g of a group G, define the conjugacy class of g to be the set $C(g) = \{xgx^{-1} : x \in G\}$.

    (i) In S_n, prove that any two r-cycles are conjugate for any $r \leq n$.

    (ii) Deduce that the number of conjugacy classes in S_n is the number $p(n)$ of ways of writing n as a sum of natural numbers.

    (Example: $3 = 3 = 2+1 = 1+1+1$ gives $p(3) = 3$
    $4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$
    gives $p(4) = 5$ etc.)